Hamiltonian Simulation: From Trotterization to Schrodingerization

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Abstract

This document rigorously reviews the mathematical frameworks and algorithms for Hamiltonian simulation in quantum computing. It begins with the fundamental Trotter-Suzuki approximation for sparse Hamiltonians, rigorously analyzing the scaling limits of higher-order expansions. It then progresses towards advanced probabilistic methods, specifically the Linear Combination of Unitary (LCU) operators and Oblivious Amplitude Amplification. Furthermore, the modern technique of Qubitization via block encoding is derived to demonstrate exact unitary mappings. Finally, the framework of Schrodingerization is introduced to simulate non-Hermitian dynamics by expanding the Hilbert space into a dilated Hamiltonian system.

Contents

1 Trotter-Suzuki Approximation

1.1 Basic Principles and Sparse Hamiltonians

The core objective of Hamiltonian simulation is to approximate the time-evolution operator $U(t) = e^{-iHt}$ for a given Hamiltonian H. When H can be decomposed into a sum of simpler, easily simulatable components, product formulas provide a straightforward algorithmic approach.

Definition 1.1 (First-Order Trotter-Suzuki Approximation)**.** Let $H = A + B$ be a Hamiltonian composed of two non-commuting Hermitian operators. The first-order Trotter-Suzuki approximation simulates the time evolution over time t by dividing it into r small time steps (where r is the Trotter number) [1]:

$$e^{-i(A+B)t} \approx \left(e^{-iA\frac{t}{r}}e^{-iB\frac{t}{r}}\right)^r \tag{1}$$

Theorem 1.1 (Error Bound of First-Order Approximation)**.** *The error ϵ induced by the first-order Trotter-Suzuki formula for $H = \sum_{j=1}^{m} H_j$ is bounded by the commutators of its constituent terms. To achieve an error at most ϵ, the required number of elementary exponential operations N_{eq} scales as [1]:*

$$N_{eq} = O\left(\frac{(mt)^2}{\epsilon}\right) \tag{2}$$

1.2 Higher-Order Trotterization and Scaling Limits

To improve the asymptotic dependence on precision ϵ, one can generalize the Trotter formula to higher orders. Suzuki introduced a recursive fractal construction to eliminate lower-order error terms mathematically.

Definition 1.2 (Higher-Order Trotter-Suzuki Formulas)**.** The symmetric second-order formula is defined as:

$$S_2(t) = e^{-iA\frac{t}{2}}e^{-iBt}e^{-iA\frac{t}{2}} \tag{3}$$

The generalized $2k$-th order formula is constructed recursively using Suzuki's fractal method:

$$S_{2k}(t) = \left[S_{2k-2}(p_k t)\right]^2 S_{2k-2}((1-4p_k)t) \left[S_{2k-2}(p_k t)\right]^2 \tag{4}$$

where the scaling factor is algebraically determined as $p_k = (4 - 4^{1/(2k-1)})^{-1}$.

Proposition 1.2 (The Inefficiency of Arbitrary High Orders and the Factor of 5)**.** *The recursive definition of $S_{2k}(t)$ strictly requires evaluating the lower-order operator S_{2k-2} exactly 5 times per single time step. Let $N_{stages}(2k)$ be the number of base S_2 segments required for a $2k$-th order approximation. The recurrence relation is trivially given by:*

$$N_{stages}(2k) = 5 \times N_{stages}(2k-2) \implies N_{stages}(2k) = 5^{k-1} \tag{5}$$

While the $2k$-th order approximation significantly suppresses the asymptotic error term to $\epsilon = O(t^{2k+1}/r^{2k})$, this fractal construction dictates that the number of required exponential evaluations blows up exponentially. The overall gate complexity to achieve a target error ϵ scales as:

$$N_{2k} = O\left(5^{k-1}\frac{t^{1+1/2k}}{\epsilon^{1/2k}}\right) \tag{6}$$

As $k \to \infty$, the asymptotic exponent $1/2k$ approaches 0, severely reducing the polynomial dependency on the precision $1/\epsilon$. However, the constant prefactor 5^{k-1} grows exponentially. For practical precision targets, increasing the order beyond $k = 1$ (2nd order) or $k = 2$ (4th order) introduces extreme gate overhead that completely outweighs the asymptotic benefits. This proves that arbitrary high-order Trotterization is fundamentally inefficient, motivating the necessity for LCU and Qubitization.

1.3 Exact Simulation of 1-Sparse Hamiltonians

To accurately simulate a realistic physical system, Hamiltonians are often represented as d-sparse matrices. Efficient simulation relies on two fundamental quantum oracles [1]:

- O_f: Calculates the column index, mapping $|x\rangle |i\rangle \to |x\rangle |f(x, i)\rangle$.
- O_H: Queries the matrix element, mapping $|x\rangle |y\rangle |0\rangle \to |x\rangle |y\rangle |H_{xy}\rangle$.

A critical building block for general simulation is the exact simulation of strictly 1-sparse blocks. Because every row and column contains exactly one non-zero element, the dynamics natively and exhaustively split into two distinct physical cases without requiring any classical branching algorithms.

1.3.1 CASE I & II: Physical Meanings

- **CASE I (Off-Diagonal, $x \neq f(x)$):** The unitary dynamics involve a non-trivial rotation and population transfer within the invariant 2D subspace uniquely spanned by $\{|x\rangle, |f(x)\rangle\}$.
- **CASE II (Diagonal, $x = f(x)$):** The state maps strictly to itself. The dynamics within this isolated 1D subspace simply apply a pure global phase $e^{-iH_{x,x}t}$ to the target state $|x\rangle$.

1.3.2 Algorithmic Implementation

Instead of relying on complex physical wire permutations which can obscure the exact mathematical logic, we can systematically describe the coherent execution of both cases using a unified algorithmic sequence. Algorithm 1 strictly outlines the exact quantum operations applied step-by-step to the joint state.

This rigorous algorithmic formulation clearly demonstrates the fundamental necessity of exactly **4 oracle queries** ($2\times O_f$ and $2\times O_H$) per simulation step. The intermediate operations dynamically project the system into either a 2×2 or 1×1 invariant subspace, allowing the exact unitary time evolution e^{-iHt} to be applied coherently across both CASE I and CASE II without any classical measurements [1, 4].

2 Linear Combination of Unitaries (LCU)

2.1 The LCU Framework

To bypass the fundamental limits of Trotterization precision, the Linear Combination of Unitary operators (LCU) method is utilized [2].

Algorithm 1 Exact Unified Simulation of a 1-Sparse Hamiltonian Block

Require: Initial state $|\psi_0\rangle = |x\rangle_{\text{sys}} |0\rangle_{\text{idx}} |0\rangle_{\text{cmp}} |0\rangle_{\text{val}}$, Simulation time t
Ensure: Final state $|\psi_{\text{final}}\rangle = e^{-iHt} |x\rangle_{\text{sys}} |0\rangle_{\text{idx}} |0\rangle_{\text{cmp}} |0\rangle_{\text{val}}$

```
Query O_f: |x>_sys |0>_idx ↦ |x>_sys |f(x)>_idx
Compute comparison c ← (x > f(x)) and store in |cmp>
Apply Controlled-SWAP on sys and idx registers controlled by |cmp>
        ▷ Registers hold |min> |max>, guaranteeing symmetry H_{x,f(x)} = H*_{f(x),x}
Query O_H: Extract matrix element into |val> register
        ▷ State becomes |min> |max> |c> |H_{x,f(x)}>
Uncompute sorting: Apply Controlled-SWAP and CMP†
        ▷ State restored to |x> |f(x)> |0> |H_{x,f(x)}>
Apply CNOT logic to evaluate parity b = x ⊕ f(x)
if b ≠ 0 then                                   ▷ CASE I: Off-diagonal element
    Apply Hadamard sequence to properly define the 2D invariant subspace
    Apply Controlled-R_x rotation parameterised by |val> for exact population transfer
else                                            ▷ CASE II: Diagonal element
    Apply diagonal phase accumulation e^{-iH_{x,x} t}
end if
Uncompute parity evaluation b via reverse CNOTs
Uncompute value register |val>:
    Apply CMP and Controlled-SWAP to re-sort indices
    Query O_H† to erase |H_{x,f(x)}> to |0>
    Apply Controlled-SWAP and CMP† to un-sort indices
Uncompute index register |idx>: Query O_f† to erase |f(x)> to |0>
return Evolved state e^{-iHt} |x>_sys ⊗ |0> |0> |0>
```

Definition 2.1 (LCU Decomposition)**.** Let A be an arbitrary bounded operator acting on a state $|\psi\rangle$. We decompose A into a linear combination of unitary operators U_j [2]:

$$A = \sum_j \alpha_j U_j \tag{7}$$

where $\alpha_j > 0$ are real coefficients. The normalization factor is defined as $\alpha = \sum_j \alpha_j$. The target state we wish to prepare is strictly proportional to $A|\psi\rangle$.

To implement this probabilistically on a quantum computer, we introduce an ancillary register initialized to $|0\rangle^{\otimes m}$ and strictly define two global operations: Prep and Select.

Definition 2.2 (Prep and Select Operators)**.** The state preparation operator Prep creates a specific superposition of ancillary states, uniquely weighted by the square roots of the coefficients α_j [2]:

$$\text{Prep}\,|0\rangle^{\otimes m} = \sum_j \sqrt{\frac{\alpha_j}{\alpha}}\,|j\rangle \tag{8}$$

The controlled unitary application operator, Select, acts as a massive multiplexer. It applies the specific unitary U_j to the system register heavily conditioned on the computational basis state of the ancilla [2]:

$$\text{Select} = \sum_j |j\rangle\langle j| \otimes U_j \tag{9}$$

Proposition 2.1 (Probabilistic LCU Implementation)**.** *By systematically applying the operational sequence* $(Prep^\dagger \otimes I)Select(Prep \otimes I)$ *to the initial joint state* $|0\rangle^{\otimes m}|\psi\rangle$*, the mathematical output yields [2]:*

$$|0\rangle^{\otimes m}\frac{A}{\alpha}|\psi\rangle + |\Phi^\perp\rangle \tag{10}$$

where $|\Phi^\perp\rangle$ *represents all orthogonal failure states wherein the ancilla is not in the zero state. Measuring the ancilla strictly in the* $|0\rangle^{\otimes m}$ *state successfully applies* A *with a success probability of* $\frac{\langle\psi|A^\dagger A|\psi\rangle}{\alpha^2}$.

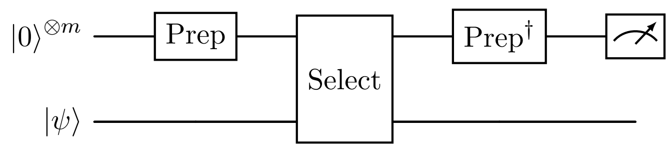

Figure 1: Standard quantum circuit for the probabilistic Linear Combination of Unitaries (LCU). The Select operator correctly spans across both registers as a unified block encoding structure [2].

2.2 Oblivious Amplitude Amplification (OAA)

Since the inherent success probability of pure LCU is strictly dependent on the input state $|\psi\rangle$, standard Quantum Amplitude Amplification (QAA) cannot be directly applied. Standard QAA requires exact knowledge of the initial state's overlap with the target subspace, which is mathematically impossible if $|\psi\rangle$ is an unknown or arbitrary quantum state. Oblivious Amplitude Amplification (OAA) cleverly circumvents this profound limitation by exploiting the unitary nature of the target operator [2].

Theorem 2.2 (Oblivious Amplitude Amplification)**.** *Let* $L = (Prep^\dagger \otimes I)Select(Prep \otimes I)$ *represent the entire LCU sequence. Suppose the operator* A *encoded by* L *is strictly proportional to a unitary operator* U*, such that* $A = \sin(\theta)U$ *for some constant angle* θ*. We define the reflection operator* $R_0 = (I - 2|0\rangle\langle 0|^{\otimes m} \otimes I)$ *over the initial zero state of the ancilla. The OAA iteration step* W *is mathematically constructed as [2]:*

$$W = -LR_0L^\dagger R_0 \tag{11}$$

Applying W *strictly rotates the state within an invariant two-dimensional subspace, boosting the amplitude of the target state deterministically without any prior knowledge of* $|\psi\rangle$*.*

Derivation of Invariant Subspace Dynamics. When the LCU operator L is applied to the initial state $|0\rangle^{\otimes m}|\psi\rangle$, the resulting state uniquely decomposes into a "good" target component and a "bad" orthogonal component. Because $A = \sin(\theta)U$ and U is mathematically unitary ($U^\dagger U = I$), the action of L is strictly given by:

$$L(|0\rangle^{\otimes m}|\psi\rangle) = \sin(\theta)|0\rangle^{\otimes m}U|\psi\rangle + \cos(\theta)|\Phi^\perp\rangle \tag{12}$$

where $|\Phi^\perp\rangle$ is a normalized error state that is strictly orthogonal to the success subspace, explicitly satisfying $(\langle 0|^{\otimes m} \otimes I)|\Phi^\perp\rangle = 0$.

The reflection operator R_0 acts exclusively on the ancilla register, inverting the phase of the zero state. Applying R_0 to the evolved state yields:

$$R_0L(|0\rangle^{\otimes m}|\psi\rangle) = -\sin(\theta)|0\rangle^{\otimes m}U|\psi\rangle + \cos(\theta)|\Phi^\perp\rangle \tag{13}$$

By applying the full iterator $W = -LR_0L^\dagger R_0$, the transformation geometrically performs a precise 2θ rotation within the invariant 2D subspace uniquely spanned by the orthonormal basis $\{|0\rangle^{\otimes m} U |\psi\rangle, |\Phi^\perp\rangle\}$.

Crucially, because U is unitary, the rotation angle 2θ depends *only* on the macroscopic LCU normalization coefficient α (where $\sin(\theta) = 1/\alpha$), and is entirely independent of the specific amplitudes of the unknown input state $|\psi\rangle$. Therefore, by applying the walk operator W approximately $\mathcal{O}(\alpha)$ times, the amplitude of the target state $|0\rangle^{\otimes m} U |\psi\rangle$ deterministically approaches unity, rendering the amplification process perfectly "oblivious" to the input data. □

3 Qubitization and Block Encoding

3.1 Mathematical Formulation of Block Encoding

Qubitization extends the LCU methodology by perfectly encoding an arbitrary, non-unitary matrix H into a larger unitary matrix U. This elegantly avoids the probabilistic failure modes of LCU by mapping the eigenvalues of H directly to quantum phases [4].

Definition 3.1 (Block Encoding)**.** A unitary matrix U acting on an extended Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_s$ is said to be a block encoding of a system Hamiltonian H (scaled by a constant α) if the top-left block of U completely contains $\frac{H}{\alpha}$. Formally, for a specific ancilla state $|0\rangle_a$ [4]:

$$(\langle 0|_a \otimes I_s) U (|0\rangle_a \otimes I_s) = \frac{H}{\alpha} \tag{14}$$

3.2 Invariant Subspace Dynamics and the Walk Operator

To deterministically extract the encoded information, Qubitization constructs a quantum walk operator W. This operator generates a rotational dynamic that completely confines the state within a highly specific, isolated two-dimensional subspace.

Theorem 3.1 (Invariant 2D Subspace under Qubitization)**.** *Let $|\lambda_i\rangle_s$ be an eigenvector of H with eigenvalue λ_i. Define the ancilla projection operator $\Pi = |0\rangle\langle 0|_a \otimes I_s$ and the reflection operator $R_0 = 2\Pi - I$. The Qubitization walk operator is defined as:*

$$W = R_0 U^\dagger R_0 U \tag{15}$$

Applying W to the initial state $|\psi_0\rangle = |0\rangle_a |\lambda_i\rangle_s$ induces a precise rotation by an angle $2\theta_i$ (where $\cos\theta_i = \lambda_i/\alpha$) within the invariant 2D subspace strictly spanned by $\{|\psi_0\rangle, U^\dagger |\psi_0\rangle\}$.

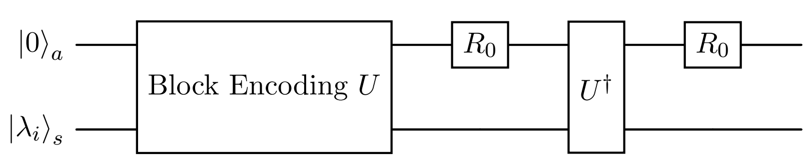

Figure 2: One complete iteration of the Qubitization walk operator $W = R_0U^\dagger R_0U$. The Block Encoding matrix U securely entangles both registers, while the reflection operator $R_0 = (2|0\rangle\langle 0|_a - I)$ isolates and reflects purely around the ancilla's zero state. This alternating sequence guarantees the rigorous preservation of the invariant subspace [4].

Rigorous Derivation of the Rotation Matrix. Let us define the orthonormal computational basis for the invariant subspace. The first basis vector is trivially the initial state: $|e_1\rangle = |\psi_0\rangle$. By the definition of block encoding, $\Pi U |\psi_0\rangle = \frac{\lambda_i}{\alpha} |\psi_0\rangle = \cos\theta_i |\psi_0\rangle$. This implies the action of the inverse unitary is $\langle\psi_0| U^\dagger |\psi_0\rangle = \cos\theta_i$. We construct an orthogonal basis vector $|e_2\rangle$ via the Gram-Schmidt process:

$$|e_2\rangle = \frac{U^\dagger |\psi_0\rangle - \cos\theta_i |\psi_0\rangle}{\sin\theta_i} \tag{16}$$

Clearly, $\langle e_1|e_2\rangle = 0$. In this orthonormal basis $\{|e_1\rangle, |e_2\rangle\}$, the state $U^\dagger |\psi_0\rangle$ is elegantly expressed as $\cos\theta_i |e_1\rangle + \sin\theta_i |e_2\rangle$.

Now, we systematically evaluate the action of $W = R_0 U^\dagger R_0 U$ on $|e_1\rangle$. First, note that $U|e_1\rangle = U|\psi_0\rangle$. Since $\Pi U |\psi_0\rangle = \cos\theta_i |\psi_0\rangle$, we can express the first reflection as $R_0 U |e_1\rangle = (2\Pi - I)U|\psi_0\rangle = 2\cos\theta_i |\psi_0\rangle - U|\psi_0\rangle$. Next, multiplying by $U^\dagger$ gives:

$$U^\dagger R_0 U |e_1\rangle = 2\cos\theta_i U^\dagger |\psi_0\rangle - I|\psi_0\rangle = 2\cos\theta_i(\cos\theta_i |e_1\rangle + \sin\theta_i |e_2\rangle) - |e_1\rangle \tag{17}$$

Finally, applying the last reflection R_0 (which simply maps $|e_1\rangle \mapsto |e_1\rangle$ and $|e_2\rangle \mapsto -|e_2\rangle$ because $|e_2\rangle$ has strictly zero overlap with $|0\rangle_a$), we obtain the final state:

$$W|e_1\rangle = (2\cos^2\theta_i - 1)|e_1\rangle + 2\sin\theta_i\cos\theta_i |e_2\rangle = \cos(2\theta_i)|e_1\rangle + \sin(2\theta_i)|e_2\rangle \tag{18}$$

This mathematically establishes that W is precisely a rotation operator by an angle $2\theta_i$ within the subspace. Consequently, the eigenvalues of W are uniquely determined and strictly bound as $e^{\pm i2\theta_i}$. □

3.3 Hamiltonian Simulation via Quantum Signal Processing

The profound significance of the walk operator W is that its eigenvalues $e^{\pm i2\arccos(\lambda_i/\alpha)}$ flawlessly encode the Hamiltonian's eigenvalues as purely unitary quantum phases.

To simulate the time evolution operator e^{-iHt}, one could ostensibly perform Quantum Phase Estimation (QPE) on W to explicitly extract θ_i and manually apply the phase. However, QPE inherently requires massive ancillary registers and computationally expensive inverse Quantum Fourier Transforms.

Qubitization elegantly bypasses this entire bottleneck by interleaving the walk operator W with parameterized ancilla rotations, a framework known as Quantum Signal Processing (QSP). By mathematically exploiting Jacobi-Anger expansions, a carefully chosen sequence of $\mathcal{O}(\alpha t + \log(1/\epsilon))$ walk steps synthesizes a precise polynomial approximation (specifically, Chebyshev polynomials of the first kind, $T_k(x)$) that directly reproduces the time evolution operator. This achieves the optimal theoretical limit for Hamiltonian simulation without any of the massive resource overheads traditionally associated with phase estimation [4].

4 Schrodingerization

4.1 Non-Hermitian Dynamics and Time-Ordered Evolution

Standard quantum mechanics relies fundamentally on unitary evolution, governed by Hermitian observables. However, many practical classical and quantum systems, such as the heat equation, Fokker-Planck equations, or open quantum systems, exhibit dissipative, non-Hermitian dynamics [3].

Consider a general linear differential equation for a quantum or classical state $|u(t)\rangle$:

$$\frac{d}{dt}|u(t)\rangle = L(t)|u(t)\rangle \tag{19}$$

If the linear operator $L(t)$ is strictly independent of time, the exact analytical solution is simply $|u(t)\rangle = e^{Lt}|u(0)\rangle$. However, when $L(t)$ varies with time, a profound algebraic complication arises: operators evaluated at different times generally do not commute, meaning $[L(t_1), L(t_2)] \neq 0$ for $t_1 \neq t_2$. Consequently, the naive exponential solution $e^{\int L(s)ds}$ becomes mathematically invalid. To resolve this fundamental issue, we must introduce the time-ordering operator $\mathcal{T}$.

Definition 4.1 (Time-Ordering Operator and Dyson Series)**.** The time-ordering operator $\mathcal{T}$ dictates that in any product of time-dependent operators, the operators must be strictly ordered from right to left in ascending order of time. For two specific times t_1 and t_2, it is mathematically defined as:

$$\mathcal{T}\{L(t_1)L(t_2)\} = \begin{cases} L(t_1)L(t_2) & \text{if } t_1 > t_2 \\ L(t_2)L(t_1) & \text{if } t_2 > t_1 \end{cases} \tag{20}$$

Using this operator, the formal analytical solution to the time-dependent differential equation is rigorously expressed as an infinite perturbation expansion known as the Dyson series:

$$\begin{aligned} |u(t)\rangle &= \left[I + \int_0^t L(t_1)dt_1 + \int_0^t \int_0^{t_1} L(t_1)L(t_2)dt_2dt_1 + \dots \right]|u(0)\rangle \\ &= \mathcal{T}\exp\left(\int_0^t L(s)ds\right)|u(0)\rangle \end{aligned} \tag{21}$$

The time-ordering operator $\mathcal{T}$ ensures that the state vector is evolved sequentially through infinitesimal time steps, strictly respecting the chronological causality of the physical system.

To embed this dissipative, time-ordered evolution into a quantum computer, we algebraically decompose the linear operator $L(t)$ into its skew-Hermitian and Hermitian components [3]:

$$L(t) = L_A(t) + L_H(t) = -iH(t) + K(t) \tag{22}$$

where $H(t) = \frac{i}{2}(L(t) - L^\dagger(t))$ and $K(t) = \frac{1}{2}(L(t) + L^\dagger(t))$ are both strictly Hermitian operators.

4.2 Moment-Fulfilling Exact Dilation Theorem

Because $K(t) \neq 0$, the operator $L(t)$ intrinsically generates non-unitary exponential decay or growth. Schrodingerization artificially expands the physical Hilbert space $\mathcal{H}_{\text{sys}}$ by taking a tensor product with an infinite-dimensional auxiliary continuous-variable space $\mathcal{H}_{\text{aux}}$ (parameterized by a continuous coordinate p).

Theorem 4.1 (Exact Dilation of Non-Hermitian Evolution)**.** *We define the dilated Hermitian Hamiltonian $\tilde{H}(t)$ acting on the joint space $\mathcal{H}_{sys} \otimes \mathcal{H}_{aux}$ as:*

$$\tilde{H}(t) = H(t) \otimes I_{aux} + K(t) \otimes P_{aux} \tag{23}$$

where $P_{aux} = -i\frac{d}{dp}$ is the standard Hermitian momentum operator. The exact non-Hermitian time evolution is perfectly recovered by projecting the dilated unitary evolution onto specifically prepared auxiliary states $|r\rangle$ and $\langle l|$:

$$\mathcal{T}\exp\left(\int_0^t L(s)ds\right) = (\langle l| \otimes I_{sys})\left[\mathcal{T}\exp\left(-i\int_0^t \tilde{H}(s)ds\right)\right](|r\rangle \otimes I_{sys}) \tag{24}$$

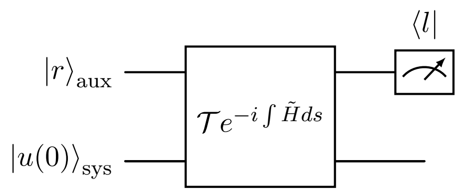

Figure 3: Conceptual quantum circuit for Schrodingerization. An auxiliary continuous variable state $|r\rangle$ is prepared. The joint system undergoes natural unitary Hamiltonian evolution under the dilated operator $\tilde{H}$. Finally, a targeted projection onto the dual state $\langle l|$ mathematically evaluates the integral, extracting the exact non-Hermitian decay dynamics without Trotter-induced norm leakage [3].

4.3 Example: Schrodingerization of the Heat Equation

To intuitively understand how the integration and momentum mapping function in practice, consider the 1D classical heat equation:

$$\partial_t u(t,x) = c\nabla^2 u(t,x) \tag{25}$$

Here, the spatial operator $L = c\nabla^2$ is purely real and negative semi-definite, meaning it is strictly Hermitian ($H = 0$) but causes exponential dissipation. We define a warped, higher-dimensional state variable $w(t,x,p)$ by introducing the auxiliary continuous variable p:

$$w(t,x,p) = e^{-p}u(t,x), \quad p > 0 \tag{26}$$

Since $\partial_p w(t,x,p) = -e^{-p}u(t,x) = -w(t,x,p)$, we can mathematically replace the dissipative operator L acting on the system with a spatial derivative acting on p:

$$\partial_t w(t,x,p) = L\left(e^{-p}u(t,x)\right) = -L\partial_p w(t,x,p) \tag{27}$$

By substituting the Hermitian momentum operator $P = -i\partial_p$, the equation rigorously transforms into:

$$\partial_t w = -L(iP)w \implies i\partial_t w = (L \otimes P)w \tag{28}$$

Since both L and P are natively Hermitian, the dilated operator $\tilde{H} = L \otimes P$ is perfectly Hermitian. The system now flawlessly obeys the standard unitary Schrödinger equation $i\partial_t |w\rangle = \tilde{H}|w\rangle$.

After simulating this unitary evolution on a quantum computer via Qubitization or LCU, the original classical solution $u(t,x)$ is exactly recovered by integrating out the auxiliary momentum space (which physically corresponds to evaluating the projection $\langle l|$ across the continuum):

$$u(t,x) = \int_0^\infty w(t,x,p)dp = \int_0^\infty e^{-p}u(t,x)dp = u(t,x) \times 1 = u(t,x) \tag{29}$$

This profound integral mapping effectively brings the entire domain of classical dissipative PDEs and non-unitary dynamics into the realm of efficient, coherent quantum simulation [3].

References

[1] Dominic W Berry, Graeme Ahsan, Richard Cleve, and Barry C Sanders. Efficient quantum algorithms for simulating sparse hamiltonians. *Communications in Mathematical Physics*, 270(2):359–371, 2007.

[2] Dominic W Berry, Andrew M Childs, Richard Cleve, Robin Kothari, and Rolando D Somma. Simulating hamiltonian dynamics with a truncated taylor series. *Physical review letters*, 114(9):090502, 2015.

[3] Shi Jin, Nana Liu, and Stanley Osher. Quantum simulation for partial differential equations via schrodingerisation. *arXiv preprint arXiv:2212.13969*, 2022.

[4] Guang Hao Low and Isaac L Chuang. Hamiltonian simulation by qubitization. *Quantum*, 3:163, 2019.